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Recovery of Liouville dynamics in quantum mechanically suppressed chaotic behaviour

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Abstract. It is shown that classical Liouville dynamics are recovered from quantum mechanically suppressed chaotic motion by introducing a measurement process into the pure dynamics. Such a process is a quantum non-demolition measurement for information on the phase space probability distribution. A fully quantum dynamical model of such a process, which is based on a von Neumann's lattice basis, is proposed. The quantum fluctuation of the measurement system releases the host system from a quantum suppression, thereby restoring an entire classical motion in the phase space.

1. Introduction

In chaotic behaviours the discrepancy between quantum and classical dynamics seems to be quite striking. One of the characteristics of this discrepancy is the shortness of the timescale for which the semiclassical description of dynamics holds correct.

Let us denote such a timescale by t^*. First we estimate t^* for non-chaotic integrable systems. We assume that the system is in a state with minimal quantum uncertainty i.e. $\Delta p \sim \Delta q \sim \sqrt{\hbar}$, where p and q are momentum and position, respectively, and $\hbar$ is the Planck constant. Classically the error in the initial state is amplified in proportion to time. Since the semiclassical description breaks down when the quantum fluctuation becomes macroscopic, we can estimate t^* to be $\hbar^{-1/2}$. This is quite a long timescale i.e. $t^* \sim 10^{13}$ s for $\hbar \sim 10^{-27}$ (in CGS units). This is not, however, the case at all for chaotic systems. For the chaotic system a small error is expanded exponentially in time with Lyapunov exponent α, and the quantum fluctuation due to the initial uncertainty is amplified as $\sqrt{\hbar}\exp(\alpha t)$. At $t^* \sim (1/2\alpha)\log(1/\hbar)$ the semiclassical description breaks down. This timescale is extremely short since it is estimated to be only a few tens of seconds!

Beyond t^* the quantum dynamics reveals its own nature: Chirikov *et al* (1981) conjectured that t^* is the timescale beyond which the chaotic diffusion becomes suppressed (Casati *et al* 1979). Indeed at t^* a transition takes place in the morphology of wavefunctions (Berry *et al* 1979), and the quantal orbital instability is observed up to a similar (but much longer, in general) timescale (Toda and Ikeda 1987a). Therefore, t^* characterises the timescale beyond which the quantum nature suppresses chaotic behaviour and makes the system 'anomalously' stable (Shepelyansky 1983, Casati *et al* 1986). However the timescale t^* is too short from the macroscopic point of view. It is unbelievable that in such a short timescale chaotic behaviour is destabilised by quantum effects. It should be shown that there exists a certain generic mechanism which destroys the quantum suppression and revives the classical chaotic dynamics.

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Recently Shepelyansky (1983) and Ott *et al* (1984) have pointed out that the classical chaotic diffusion is recovered by applying small dynamical perturbations. This phenomenon is quite interesting in view of the quantum-classical correspondence. Their studies, however, are concerned only with the recovery of chaotic diffusion in *momentum space*. What we would like to elucidate is how the classical chaotic motion is recovered entirely in the *phase space* where the information necessary and sufficient for describing the classical motion is provided.

The aim of the present paper is to show that the quantum suppression of chaotic motion is removed and classical motion is recovered in the phase space quite naturally by introducing a process of measurement. Such a process is a quantum non-demolition measurement for physical quantities needed for describing the classical dynamics. To be concrete, in the present paper we specifically consider the chaotic diffusion phenomenon extensively studied by Casati *et al* (1979) because it most definitely manifests the discrepancy between the quantum and the classical dynamics. However, we believe that the results mentioned below would be quite general.

The outline of the present paper is as follows. In § 2 we reconfirm numerically the existence of an anomalous timescale similar to t^* in the chaotic diffusion phenomenon. We consider the significance of such a timescale by observing the time evolution of the wavepacket in the phase space. On the basis of such an observation we propose in § 3 an 'ideal' perturbation process which may help the system to restore the semiclassical dynamics in the phase space. Further, we show that such a process can be interpreted as a back-action due to a measurement of the information on the probability distribution in the phase space. A fully quantum dynamical model of measurement process which gives rise to the same effect as the 'ideal' perturbation is proposed. In § 4, we investigate numerically the effect of the ideal perturbation process. Various evidence showing that the evolution process into which the measurement process is incorporated restores classical Liouville dynamics in the phase space is presented. In § 5 we discuss the significance of incorporating the measurement process into the whole quantum dynamics.

2. Existence of an anomalous timescale

The system we consider in the present paper is the kicked rotor described by the Hamiltonian

$$H = \tfrac{1}{2}\hat{p}^2 + K \sum_{n=-\infty}^{+\infty} \delta(t-n)\, V(\hat{\theta}) \tag{2.1}$$

where $\hat{p} \equiv -\mathrm{i}\hbar\partial/\partial\hat{\theta}$ and $\hat{\theta}$ are the momentum and position operators, respectively, and $V(\hat{\theta})$ is a periodic function with period 2π. Here, we choose

$$V(\hat{\theta}) = \cos\hat{\theta}. \tag{2.2}$$

The classical motion of (1) is described by the well known standard mapping:

$$\theta_{t+1} = \theta_t + p_{t+1} \qquad p_{t+1} = p_t - KV'(\theta_t) \tag{2.3}$$

where θ_t and p_t are the angle and momentum, respectively, at the tth step. The motion described by the mapping rule (2.3) becomes globally chaotic as the parameter K exceeds the critical value $K_c = 0.971\ldots$, and a diffusion occurs across the momentum space (Chirikov 1979, Greene 1979). Hence the second-order moment M of momentum increases linearly in time, i.e. $M \equiv \langle (p - \langle p \rangle)^2 \rangle = D_{\mathrm{CL}} t$, where D_{CL} is the classical

diffusion constant. In quantum dynamics an iteration corresponding to the mapping (2.1) is achieved by operating the unitary operator

$$U = \mathcal{T} \exp\left(-i \int_0^1 H(t')\, dt'/\hbar\right) = \exp\left(-i \frac{\hbar}{2} \frac{\partial^2}{\partial \hat{\theta}^2}\right) \exp\left(-\frac{iK}{\hbar} V(\hat{\theta})\right) \tag{2.4}$$

on the wavefunction $\psi(\hat{\theta})$. Here $\mathcal{T}$ is the time-ordering operator and the periodic boundary condition

$$\psi(\hat{\theta} + 2\pi) = \psi(\hat{\theta})$$

is assumed for $\psi(\hat{\theta})$.

Quantal motion also exhibits a diffusion phenomenon. However, the time regime in which the diffusive motion is observed is quite restricted. In figure 1(a) we show a typical behaviour of M obtained starting from a momentum eigenstate. There are three time regimes. In the first regime $t < T_r$ the second-order moment M increases in agreement with the classical chaotic diffusion. A significant deviation from the classical motion emerges in the second regime $T_r < t < T_s$, and M reaches a saturation level in the third regime $t > T_s$. The origin of saturation in the third regime is fully quantum mechanical, being understood in connection with Anderson localisation (Fishman *et al* 1982). The Anderson localisation is quite unstable against time-

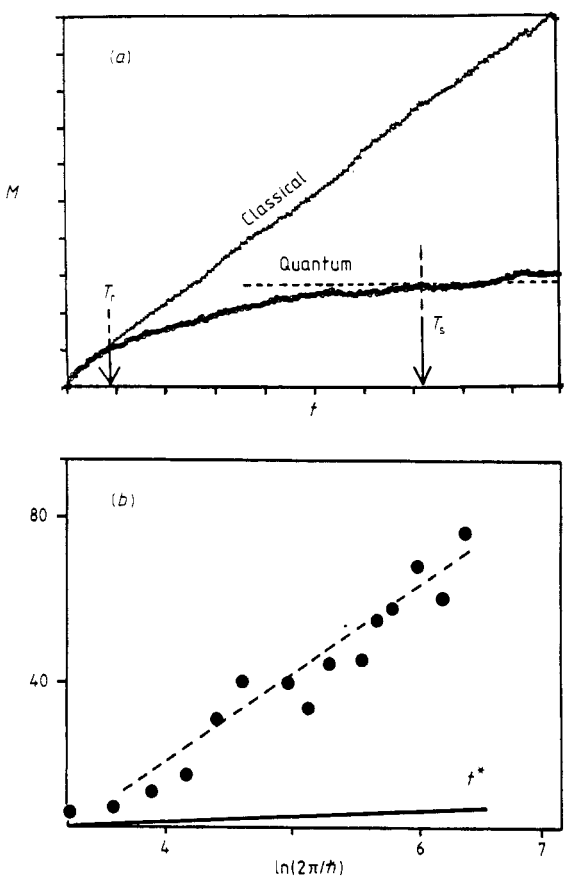

Figure 1. (a) Time evolution of the second-order moment M for classical and quantum kicked rotors. (b) Dependence of the timescales T_r and t^* upon $\hbar^{-1}$. $K = 2.0$.

dependent perturbation (Thouless 1977). This provides an important implication for the release of the system from the Anderson localisation. Indeed, the classical diffusion can be recovered by applying an appropriate external noise (Ott *et al* 1984). However, it is not very clear whether such a perturbation restores the classical chaotic dynamics in the phase space or not. As will be noted later, a recovery of the classical-like diffusion does not necessarily mean a restoration of the classical phase space motion.

In view of quantum-classical correspondence, there is no problem for the second timescale T_s, because it goes to infinity more promptly than $\hbar^{-2}$ as $\hbar$ approaches zero (Chirikov *et al* 1981, Sheplyansky 1983). The problem is the first timescale T_r: T_r also increases as $\hbar$ goes to zero, but its increase is too slow. In figure 1(b) we show how the numerically computed T_r depends upon $\hbar$. It seems to have quite similar dependence on $\hbar$ as does t^*, i.e. $T_r \propto \log 1/\hbar$. Existence of such behaviour is confirmed up to $\hbar \sim 10^{-3}$ but there is, of course, no evidence that such behaviour persists up to the actual value $\hbar \sim 10^{-27}$. However, if this is the case, there appears a paradoxical fact: T_r is as short as $t^* \sim$ a few tens of seconds ($\hbar \sim 10^{-27}$) and this is a timescale on which we believe that the classical description works well. On such a classical timescale we could observe the quantum suppression of chaotic diffusion! The implausibility of the paradox implies that there exists a serious gap between the classical chaotic dynamics and the quantal chaotic dynamics.

The first timescale T_r has a quite similar dependence upon $\hbar$ as does t^*, but the former is more than 10 times longer than the latter. We first elucidate the meaning of T_r, which may provide some hints on a possible mechanism to be introduced for removing the discrepancy between the quantal dynamics and the classical one. To this end we observe the motion of the wavefunction mapped onto the c-number phase space (Takahashi and Saito 1985). Let us introduce a phase space representation (quasiprobability) of the density operator $\rho \equiv |\psi(t)\rangle\langle\psi(t)|$ which is defined by

$$Q(\theta, p, t) \equiv \langle\langle \theta, p|\rho(t)|\theta, p\rangle\rangle \tag{2.5}$$

where $|\theta, p\rangle\rangle$ is the coherent state with respect to the annihilation operator $\hat{a} \equiv (\hat{\theta} + i\hat{p})/\sqrt{2\hbar}$ generated from the vacuum state $|\text{vac}\rangle$ of $\hat{a}$:

$$|\theta, p\rangle\rangle \equiv \exp\{(\theta + ip)\hat{a}^+/\sqrt{2\hbar} - (\theta - ip)\hat{a}/\sqrt{2\hbar}\}|\text{vac}\rangle.$$

$Q(\theta, p, t)$ is a quantal counterpart of the classical probability distribution function defined in the phase space. We observe the motion of a contour line Σ_P of $Q(\theta, p, t)$. Σ_P is chosen in such a way that the total probability in the region σ_P enclosed by Σ_P is a constant value $P(\leqslant 1)$ (Toda and Ikeda 1987a). A typical example showing a time evolution of contour lines of $Q(\theta, p, t)$ is shown in figure 2. We start with a coherent state localised in the phase space (figure 2(a)). As time elapses, the contour lines are stretched and folded in a quite similar manner as we often experience in classical chaotic dynamics (figure 2(b)). In classical chaotic dynamics the stretching and folding operations are repeated without limit, and arbitrary fine structures can be formed in the phase space. As a result, some of the stretched parts are pushed out of the initial region and eventually give rise to the classical diffusion in momentum space. The quantum time evolution is, however, quite different from the classical evolution. When the stretched and folded structure is formed in the phase space, a quantum interference between portions of the wavepacket on nearby branches takes place. This interference forms lumps of size $O(\sqrt{\hbar})$ (Toda and Ikeda 1987b) (see figure 2(c)), which destroy the finely stretched and folded structure needed for pushing the probability out of the region in which the system is initially prepared (figure 2(d)). This is the mechanism of quantum suppression of diffusion.

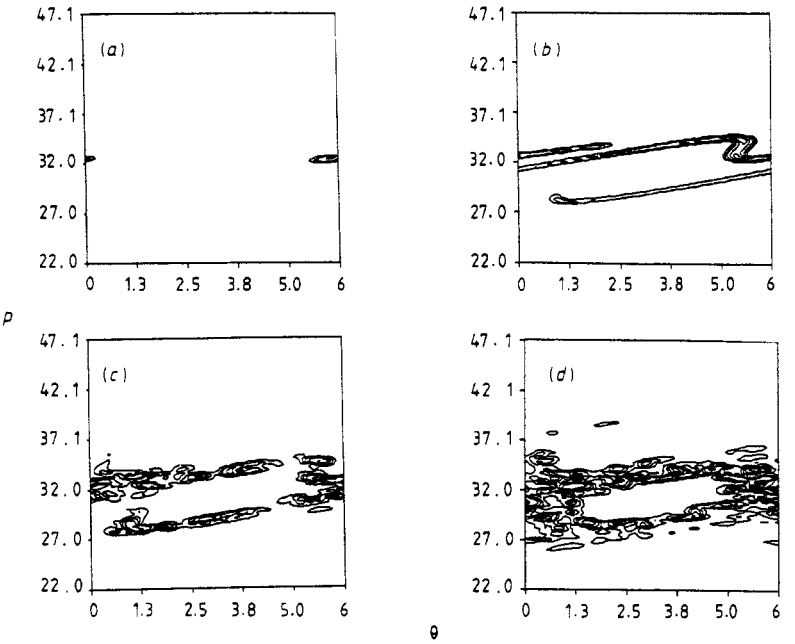

Figure 2. Time evolution of the quasiprobability distribution function $Q(\theta, p, t)$; (a) $t=1$, (b) $t=4$, (c) $t=7$, (d) $t=13$, where $K=2.0$.

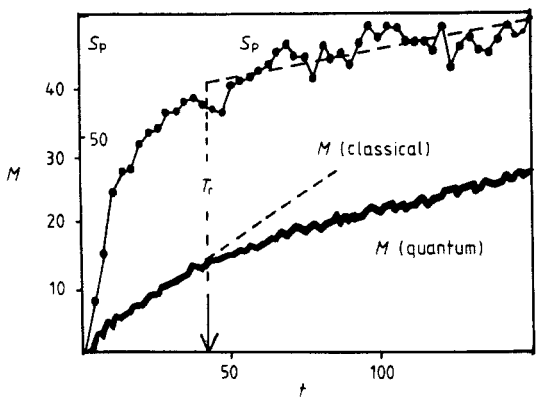

Figure 3. Relation between the time evolution of M and S_P. Here S_P is the area of the region enclosed by the contour curve of $Q(\theta, p, t)$ (see text).

The relationship between the quantum interference and the suppression of diffusion is more directly seen in figure 3. In this figure we show the evolution of the area S_P of the region σ_P in comparison with the evolution of the second-order moment M. In classical chaotic dynamics, the length of a region in the phase space increases as $L \sim e^{\alpha t}$ (α: Lyapunov exponent), whereas the thickness of the region decreases like $l \sim e^{-\alpha t}$ because the area of the region is conserved. In quantum dynamics, however, there is no scale less than $\sqrt{\hbar}$, and the thickness l ceases to decrease at $\sqrt{\hbar}$. Thenceforth, the area S_P of the region σ_P enclosed by a contour line Σ_P increases in proportion to $lL \sim \sqrt{\hbar}\, e^{\alpha t}$. Thus σ_P soon fulfils the available region in the phase space. As shown

in figure 3, it is just at this time that the mechanism of the quantum suppression begins to work significantly†.

At T_r the wavepacket which has been stretched and folded in the phase space is 'packed like sardines' in the available region in the phase space. Everywhere in this region interference between neighbouring parts of the stretched and folded wavepacket occurs and a systematic emission of probability out of this region is inhibited. Therefore, if we introduce some natural mechanism which destroys the quantum coherence everywhere in the phase space, the quantum interference is suppressed and a classical chaotic motion in the phase space may be restored. This will be discussed in the next section.

3. Phase randomisation on the von Neumann lattice measurement of phase space information

Physically the semiclassical limit means not only to take the mathematical limit $\hbar \to 0$. It also practically means to make the motion contributing to the chaotic behaviour macroscopic. In a macroscopic system, a certain collective mode (e.g. the centre-of-mass coordinates) will contribute to the macroscopic chaotic motion. However, there exist a number of degrees of freedom which do not contribute to the macroscopic behaviour but disturb the macroscopic chaotic motion at least on microscopic scales. Such a microscopic Brownian motion leads to a diffusion of macrovariables in the phase space. Here we discuss the effect of microscopic Brownian motion briefly. Let τ_M be the characteristic timescale of macrovariables ($\tau_M = 1$ for our kicked rotor). 'Microscopic' means that the characteristic lengths of diffusion during τ_M, which we denote by $\Delta\theta(\tau_M)$ and $\Delta p(\tau_M)$ for θ and p, respectively, are of microscopic sizes. Once microscopic diffusion occurs, its reaction destroys the coherence of the wavefunction on the scales $\Delta\theta_c \sim \hbar/\Delta p(\tau_M)$ and $\Delta p_c \sim \hbar/\Delta\theta(\tau_M)$ respectively, in the phase space. Such a mechanism may destroy the quantum interference developed by quantum 'chaotic' dynamics everywhere in the available region in the phase space, and it may recover the classical chaotic dynamics, if both $\Delta\theta_c$ and Δp_c may be made microscopic. Model processes describing microscopic Brownian motion mentioned above can actually be constructed. In the present paper, however, we will introduce a more 'idealised' process. By 'ideal' we mean that the process destroys the coherence of the wavefunction everywhere in the phase space while keeping the lengths of diffusion within the minimum possible scales, i.e. $\Delta\theta(\tau_M) \sim \Delta p(\tau_M) \sim \sqrt{\hbar}$. Such a process breaks the phase space up into minimum uncertainty cells of size $\Delta\theta_c \sim \Delta p_c \sim \sqrt{\hbar}$ between which no phase correlation exists.

Here we introduce an 'ideal' process which we call the phase randomisation on the von Neumann lattice. The physical significance of the model process will be discussed in detail in the latter half of this section. This process is described by using a complete set of basis functions, each of which is localised inside of a cell of the von Neumann lattice (VNL) (von Neumann 1932), that is a square lattice in the phase space with cell size $\sqrt{h}$ ($h = 2\pi\hbar$). Let $|\theta_l, p_m\rangle$ be a basis localised in a lattice cell centred at $(\theta_l, p_m) = (\sqrt{h}\,(l+\frac{1}{2}), \sqrt{h}\,(m+\frac{1}{2}))(l, m$: integers) and assume the set $\{|\theta_l, p_m\rangle\}$ to be complete and orthogonal. Then the 'ideal' process is defined by the unitary transforma-

† Fishman *et al* (1987) predicted another dependence of T_r on $\hbar$, i.e. $T_r \propto \hbar^{-1/\gamma}(\gamma \sim 3.0)$. However, their prediction and the physical picture behind it are applicable only in the vicinity of $K = K_c$.

tion $R(t)$:

$$R(t)|\psi(t)\rangle = \sum_{l,m} e^{i\nu_{lmt}} \langle \theta_l, p_m|\psi(t)\rangle|\theta_l, p_m\rangle \tag{3.1}$$

where the perturbation strength ν_{lmt} is a random variable uncorrelated at different sites as well as at different steps, and it is characterised by the statistical average $\langle\langle \nu_{lmt}\nu_{l'm't'}\rangle\rangle = \delta_{ll'}\delta_{mm'}\delta_{tt'}\nu^2$. Von Neumann first proposed a set of displaced coherent states localised at (θ_l, p_m) as the basis function $|\theta_l, p_m\rangle$. This set of basis functions is complete, but unfortunately it is not orthogonal. A desired complete orthogonal set can be constructed by using an 'Anderson localisation technique' in the phase space. However, the following complete orthogonal basis is sufficient for practical purposes:

$$|\theta_l, p_m\rangle = \sqrt{1/L} \sum_{k=mL+1}^{m(L+1)} e^{-ik\theta_l}|k\rangle \tag{3.2}$$

where L is an integer satisfying the condition $(2\pi/L)^2 = h$ and $|k\rangle = \exp(ik\hat{\theta})/\sqrt{2\pi}$ (k: integer) is the eigenfunction of $\hat{p}$. We show in figure 4 the contour plot of the quasiprobability distribution function for the von Neumann lattice (VNL) basis (3.2). Certainly each of these bases is localised inside a VNL cell.

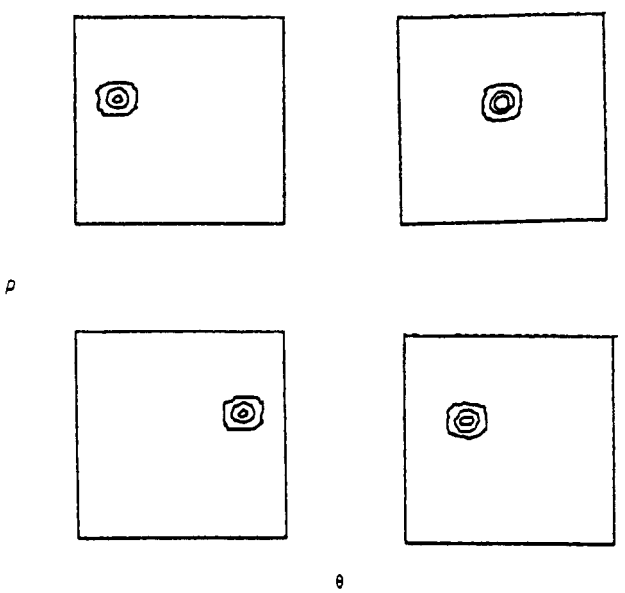

Figure 4. Contour plots of the quasiprobability distribution function for the VNL basis states defined by (3.2).

The VNL basis can be looked upon as the fundamental basis for a semiclassical simultaneous measurement of momentum and position. Indeed, using the VNL basis we can construct approximate position and momentum operators which commute with each other:

$$\tilde{\theta} \equiv \sum_{lm} \theta_l |\theta_l, p_m\rangle\langle\theta_l, p_m|$$

$$\tilde{p} \equiv \sum_{lm} p_m |\theta_l, p_m\rangle\langle\theta_l, p_m|.$$

Since the VNL bases are localised in a region of size $\sqrt{h}$ in the phase space, they approximate the true position and momentum operators $\hat{\theta}$ and $\hat{p}$ to within an accuracy of $O(\sqrt{h})$. We can thus in principle do an approximate simultaneous measurement on the basis of the VNL basis. Therefore, the VNL basis is a natural basis set for elucidating the relationship between the classical dynamics and the quantal dynamics by incorporating the semiclassical simultaneous measurement processes.

In the following, we will show that the phase randomisation on the VNL described by (3.1) can be interpreted as a back-action due to a measurement of information on the probability distribution in the phase space. Based on the VNL basis we can construct a fully quantum dynamical model of measurement process as explained below.

(1) The measurement system is composed of a number of groups of 'atoms'. The 'atoms' are excited in interaction with the host system. We can read out the information on the local probability distribution in the phase space of the host system from the total number of excited atoms in a specific group.

(2) We can count the total number of excited atoms belonging to each group. An 'atom' is modelled by a two-level system having a ground state $|+\rangle$ and an excited state $|-\rangle$.

(3) The phase space of the host system is decomposed into coarse-grained domains of almost the same semimacroscopic size, and the system interacts selectively with a specific group (say α) of atoms when the system is in a coarse-grained domain (say D_α). Each of the groups of 'atoms' has a one-to-one correspondence with each of the coarse-grained domains in the phase space of the host system. It is desired that the measurement process do not disturb the probability distribution itself; in other words, the measurement process is required to be a quantum non-demolition measurement for the local probability distribution in the phase space. This requirement restricts the form of interaction between the 'atoms' and the host system. A plausible model of the interaction Hamiltonian is

$$H_I = \sum_\alpha \zeta^{(\alpha)} \hat{R}_x^{(\alpha)} \hat{\Lambda}^{(\alpha)} \tag{3.3}$$

where $\hat{R}_x^{(\alpha)}$ stands for the collective creation operator of the two level atoms in the group α:

$$\hat{R}_x^{(\alpha)} \equiv \sum_{i=1}^{N_\alpha} \hat{\sigma}_x^{(i,\alpha)} \tag{3.4}$$

with

$$\hat{\sigma}_x^{(i,\alpha)} = |+\rangle_{(i,\alpha)}\langle -|_{(i,\alpha)} + \text{HC}. \tag{3.5}$$

$\sigma_x^{(i,\alpha)}$ is the Pauli spin matrix representing the ith atom in the group α. On the other hand $\hat{\Lambda}^{(\alpha)}$ is the projection operator of the host system, i.e.

$$\hat{\Lambda}^{(\alpha)} \equiv \sum_{(\theta_l, p_m) \in D_\alpha} |\theta_l, p_m\rangle\langle\theta_l, p_m|.$$

$\zeta^{(\alpha)}$ is the coupling constant between the host system and the 'atoms'. It is easy to see that the interaction does not change the local probability distribution in the phase space, i.e. $|\langle\theta_l, p_m|\psi\rangle|^2$.

(4) The 'atoms' are initially all in the ground states. They interact with the host system during a very short finite period T. After the interaction we can read out the information on the local probability distribution in the phase space. We assume that

T is so short that the evolution of host system by its own dynamics can be neglected. Then the final state of the composed host+measurement (='atoms') system after the interaction is described by the S matrix:

$$S = \exp -\frac{\mathrm{i}H_I T}{\hbar}$$
$$= \sum_\alpha \hat{\Lambda}_\alpha \exp -\mathrm{i}\theta^{(\alpha)} \hat{R}_x^{(\alpha)} \tag{3.6}$$

where $\theta^{(\alpha)} \equiv \zeta^{(\alpha)} T/\hbar$. Let ψ_i be the initial state before the measurement is done:

$$\psi_i = \sum_{lm} C_{lm} |\theta_l, p_m\rangle \otimes \prod_{\alpha,i} |-\rangle_{(i,\alpha)}. \tag{3.7}$$

Then the total number of excited 'atoms' in the group α is easily computed:

$$M_\alpha = \left\langle \psi_i S^+ \left| \sum_{i=1}^{N_\alpha} \right| + \right\rangle_{(i,\alpha)} \langle + |_{(i,\alpha)} |S\psi_i\rangle$$
$$= \sum_{(\theta_l, p_m) \in D_\alpha} |C_{lm}|^2 N_\alpha \sin^2 \theta^{(\alpha)} \tag{3.8}$$

where N_α is the total number atoms in the group α. Thus we can read out the coarse-grained probability distribution in the phase space $\Sigma_{(\theta_l, p_m) \in D_\alpha} |C_{lm}|^2$.

(5) If we wish to make successive measurements in an appropriate interval, a new measurement system is prepared each time a new measurement is done.

Our process does not change the probability at any VNL lattice cell, i.e. $|C_{lm}|^2$. Instead, it greatly disturbs the phase information of the complex probability amplitude. Let $\bar{C}_{lm}$ be the probability amplitude of the final state, i.e. $\bar{C}_{lm} = \langle \theta_l, p_m | \psi_f \rangle$. Then we obtain

$$\bar{C}_{lm} = C_{lm} \exp[-\mathrm{i}\theta^{(\alpha_{lm})} \hat{R}_x^{(\alpha_{lm})}] \prod_{\alpha,i} |-\rangle_{(\alpha,i)} \tag{3.9}$$

where α_{lm} denotes the domain to which the VNL state (l, m) belongs. The phase factor $\exp[-\mathrm{i}\theta^{(\alpha)} \hat{R}_x^{(\alpha)}]$ contains the operator $\hat{R}_x^{(\alpha)}$, and a fully quantum analysis is required for examining its effect.

We introduce the creation and annihilation operators

$$\hat{a}_\alpha^+ \equiv \frac{1}{\sqrt{N_\alpha}} \sum_{i=1}^{N_\alpha} |+\rangle_{(i,\alpha)} \langle -|_{(i,\alpha)} \qquad \hat{a}_\alpha \equiv \frac{1}{\sqrt{N_\alpha}} \sum_{i=1}^{N_\alpha} |-\rangle_{(i,\alpha)} \langle +|_{(i,\alpha)}. \tag{3.10}$$

To simplify the problem, we assume that the total number of excited 'atoms' is much smaller than the total number of 'atoms', i.e. $M_\alpha \ll N_\alpha$. Then the commutation relation

$$[\hat{a}_\alpha, \hat{a}_\alpha^+] = \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} (|-\rangle_{(i,\alpha)} \langle -|_{(i,\alpha)} - |+\rangle_{(i,\alpha)} \langle +|_{(i,\alpha)}) \tag{3.11}$$

is approximated by 1, and $\hat{a}_\alpha$ and $\hat{a}_\alpha^+$ can be regarded as boson annihilation and creation operators. Hence the ground state $\prod_{i=1}^{N_\alpha} |-\rangle_{(i,\alpha)}$ is the vacuum state $|\mathrm{vac}\rangle_\alpha$ of $\hat{a}_\alpha$ and

$$\exp(-\mathrm{i}\theta^{(\alpha)} \hat{R}_x^{(\alpha)}) \prod_{i=1}^{N_\alpha} |-\rangle_{(\alpha,i)} = \exp -\mathrm{i}\xi^{(\alpha)} (\hat{a}_\alpha + \hat{a}_\alpha^+) |\mathrm{vac}\rangle_\alpha \tag{3.12}$$

is no more than the coherent state of the boson operator $\hat{a}_\alpha$, where

$$\xi^{(\alpha)} \equiv \theta^{(\alpha)} \sqrt{N_\alpha} = T\zeta^{(\alpha)} \sqrt{N_\alpha}/\hbar.$$

We evolve the host system by its own dynamics by inserting measurement processes in appropriate time intervals. Assume that the evolution of the host system by its own dynamics is described by the unitary matrix V such as U (or its product). The tth step complex probability amplitude is obtained iteratively:

$$C_i(t) = \sum_j V_{ij}(t-1) C_j(t-1). \tag{3.13}$$

Here i stands for a VNL state specified by (l, m). We insert measurements between such pure evolutions and examine their effect on the whole dynamics. For the sake of simplicity we first consider a single step of measurement followed by a pure evolution process. After a single step evolution the ii' component of the density operator for the host plus measurement system is

$$\begin{aligned} &C_i(1)C_{i'}^*(1) \\ &\quad = \sum_{jj'} V_{ij}(0)\, V_{i'j'}(0)^* C_j(0) C_{j'}^*(0) |-\mathrm{i}\xi^{(\alpha_j)}\rangle\!\rangle_{\alpha_j} \\ &\qquad \times \prod_{\beta \neq \alpha_j} |\mathrm{vac}\rangle_\beta \prod_{\beta \neq \alpha_{j'}} {}_\beta\langle\!\langle \mathrm{vac}| -\mathrm{i}\xi^{(\alpha_{j'})}|_{\alpha_{j'}} \end{aligned} \tag{3.14}$$

where α_j indicates the group of 'atoms' which interact with the domain D_{α_j} containing the state j, and $|\lambda\rangle\!\rangle_\alpha$ is the coherent state $\exp(\lambda \hat{a}_\alpha^+ - \lambda^* \hat{a}_\alpha)|v\rangle_\alpha$ of the boson operator $\hat{a}_\alpha$.

Now we take the partial trace with respect to all the states of 'atoms'. Since

$$\operatorname*{Tr}_{\text{'atoms'}}(\ldots) = \int \prod_\beta \langle\!\langle \lambda_\beta|_\beta (\ldots) \prod_\beta |\lambda_\beta\rangle\!\rangle_\beta \prod_\beta \frac{\mathrm{d}^2\lambda_\beta}{\pi} \tag{3.15}$$

the partial trace over the 'atoms' states yields

$$\begin{aligned} &\operatorname*{Tr}_{\text{'atoms'}} C_i(1) C_{i'}^*(1) \\ &\quad = \sum_{jj'} \int V_{ij}(0)\, \mathrm{e}^{-\mathrm{i}\nu_{\alpha_j}} V_{i'j'}(0)\, \mathrm{e}^{\mathrm{i}\nu_{\alpha_{j'}}} C_j(0) C_{j'}^*(0) \\ &\qquad \times \mathrm{e}^{-(\nu_{\alpha_j}^2 + \nu_{\alpha_{j'}}^2)/\xi^2}\, \mathrm{d}\nu_{\alpha_j}\, \mathrm{d}\nu_{\alpha_{j'}} / \pi\xi^2. \end{aligned} \tag{3.16}$$

For simplicity we have assumed all ξ_α are equal to ξ and used the formula:

$$\begin{aligned} \langle\!\langle \mu|\lambda\rangle\!\rangle &= \exp\left(-\frac{|\mu|^2}{2} - \frac{|\lambda|^2}{2} + \mu^*\lambda\right) \\ \langle\!\langle \lambda|\mathrm{vac}\rangle\!\rangle &= \exp\left(-\frac{|\lambda|^2}{2}\right). \end{aligned} \tag{3.17}$$

Extension to a multiple-step process is straightforward because at each step the host system interacts with a renewed measurement system. Equation (3.16) means that a time-evolved behaviour can be made up by collecting all the sample processes, each of which is generated by the following successive transformations.

(1) Multiply by the random phase factor:

$$\bar{C}_i(t-1) = \mathrm{e}^{\mathrm{i}\nu_{\alpha_i}^{(t-1)}} C_i(t-1). \tag{3.18a}$$

(2) Evolve the host system according to its own dynamics:

$$C_i(t) = \sum_j V_{ij}(t-1) \bar{C}_j(t-1). \tag{3.18b}$$

The first process corresponds to the back-action due to the measurement process. After repeating these steps we obtain $C_i(t)$, which is a functional of random variables $\{\nu_\alpha^{(0)}\}\ldots\{\nu_\alpha^{(t-1)}\}$. We can compute the ii' component of the density operator of the host system by averaging over the random variables $\{\nu_\alpha^{(s)}\}_{s=0,\ldots,t-1}$ with the Gaussian statistical weight:

$$\int \exp\left(-\sum_{s,\alpha} \nu_\alpha^{(s)2}\xi^{-2}\right) C_i(t)C_{i'}^*(t) \prod_{\alpha,s} (\mathrm{d}\nu_\alpha^{(s)}/\sqrt{\pi}\xi). \tag{3.19}$$

The stochastic process (3.18a) agrees entirely with the phase randomisation on the VNL if D_α is replaced by the VNL cell. Thus the phase randomisation on the VNL represents a sample process describing a back-action due to a measurement of the phase space probability distribution. The phase $\nu_\alpha^{(s)}$ can be interpreted as the quantum fluctuation of the ground state of the 'atoms' system.

4. Numerical simulation

In this section we numerically investigate how the quantal time evolution process into which the process of measurement is incorporated modifies the pure quantum motion of the kicked rotor in the phase space. To be realistic, the size of the coarse-grained domains D_α should be made semimacroscopic; in other words, D_α should contain a large number of the VNL cells. In numerical simulation it is, however, practically impossible to make D_α sufficiently large because the available size of the VNL cell is quite restricted ($h \geqslant (4\times 10^2)^{-1}$). Therefore, we take the VNL cell itself as D_α. The process we examine in the numerical simulation is, therefore, the composite process

$$\mathrm{O}(t, \boldsymbol{\nu}^{(t)}) = UR(\boldsymbol{\nu}^{(t)}) \tag{4.1}$$

containing a set of random phases $\boldsymbol{\nu}^{(t)} \equiv \{\nu_{lm}^{(t)}\}$. Note that a solution $\psi(t) = \Pi_{s=1}^{t} \mathrm{O}(t, \boldsymbol{\nu}^{(s)})\psi(0)$ represents a sample solution for a given sequence of random variables $(\boldsymbol{\nu}^{(1)}, \ldots, \boldsymbol{\nu}^{(t)})$, which should be averaged over with the Gaussian statistical weight (see (3.19)):

$$P(\nu^{(1)}, \ldots, \nu^{(t)}) = \exp\left(-\sum_{s=0}^{t}\sum_{lm} \nu_{lm}^{(s)2}\nu^{-2}\right). \tag{4.2}$$

Henceforth ν stands for the back-action parameter $\xi \equiv T\zeta^{(\alpha)}\sqrt{N_\alpha}/\hbar$.

We carried out numerical simulations with computer-generated Gaussian noise. The diffusion process, which is suppressed in quantum dynamics, is always recovered irrespective of the magnitude of the back-action parameter ν, and the diffusion constant D_α exhibits a characteristic variation with an increase in ν. In figure 5 we depict the diffusion constant D_Q as a function of ν. For finite $\hbar$ there are two characteristic values of ν: (1) for $0 < \nu < \nu_1$; the diffusion constant $D_\mathrm{Q}(\nu)$ increases gradually toward D_CL; (2) for $\nu_1 < \nu < \nu_2$; $D_\mathrm{Q}(\nu)$ agrees with D_CL; (3) for $\nu > \nu_2$ $D_\mathrm{Q}(\nu)$ increases again and eventually reaches a limit $D_\mathrm{Q}(\infty)$. ν_1 is the least value of ν at which the classical diffusion is recovered and is closely related with the timescale T_r. It is estimated as follows. The total random phase accumulated during the period τ is estimated to be $\nu_\mathrm{tot} \sim \nu\sqrt{\tau}$, and if ν is more than $2\pi/\sqrt{T_\mathrm{r}}$, the phase accumulated at a von Neumann lattice site amounts to more than 2π before the quantum suppression mechanism begins to work. Thus $\nu_1 \sim 2\pi/\sqrt{T_\mathrm{r}}$.

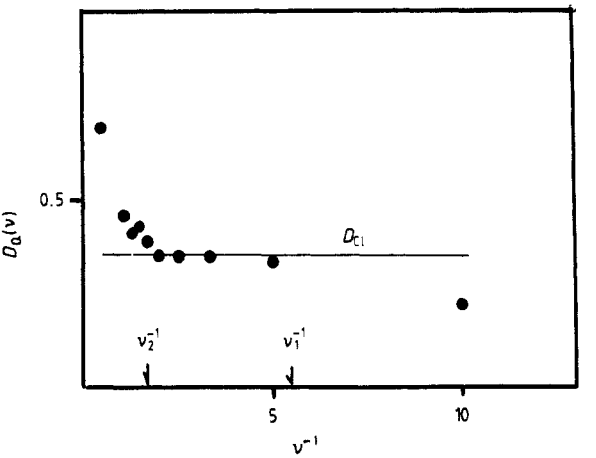

Figure 5. Dependence of the quantal diffusion constant $D_Q(\nu)$ upon the back-action parameter ν.

Above ν_2 the VNL randomisation process creates classical noise which causes D_Q to increase again. However, such an increment decreases with a decrease in $\hbar$. In figure 6 we show how $D_Q(\infty) - D_{CL}$ decreases with $\hbar$. Evidently $D_Q(\infty)$ converges to D_{CL} nicely as

$$D_Q(\infty) - D_{CL} = \gamma\hbar \tag{4.3}$$

where γ is a constant ($\gamma \simeq 7.0$ for $K = 2.0$). Thus in the small limit of $\hbar \to 0$, the classical diffusion is recovered in a considerably wide range of perturbation strength, i.e. $\nu_1 \sim T_r^{-1/2} \sim (|\log \hbar|)^{-1/2} < \nu < +\infty$. We note that the diffusion behaviour is clearly observed even in a single-sample process.

The phase randomisation on the VNL can be represented by the interaction Hamiltonian

$$H_I(\nu) = \sum_{lm} \omega_{lm} |lm\rangle\langle lm|$$

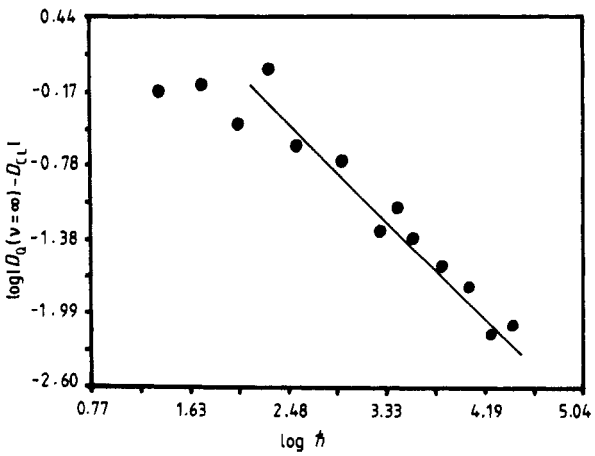

Figure 6. Convergence of $D_Q(\nu = \infty)$ to D_{CL} as $\hbar$ is decreased.

where ω_{lm} is related to ν as $\omega_{lm}T/\hbar = \nu_{lm}$ (T is the period for which the interaction is applied). Such an interaction has no classical counterparts because the characteristic length of interaction over the phase space is the minimal length of quantum uncertainty, i.e. $\sqrt{\hbar}$. This is the reason why the quantum diffusion rate for $\hbar \to 0$ approaches the classical value even for the 'wild' limit of perturbation strength, i.e. $\omega_{lm}(\nu_{lm}) \to \infty$. (If the perturbation externally introduced has a classical counterpart, D_Q for $\hbar \to 0$ should agree with the classical diffusion rate of the perturbed system, which is much larger than that of the unperturbed system.)

Although the phase randomisation on the VNL has no classical counterpart, the increment of $D_Q(\infty)$ from D_{CL} for a finite $\hbar$ implies that the phase randomisation creates an equivalent classical noise. This is quite natural because the phase randomisation on the VNL would classically introduce an uncertainty of $O(\sqrt{\hbar})$ in the phase space. Indeed the behaviour of (4.3) can be well simulated by adding classical noise of $O(\sqrt{\hbar})$ to the classical mapping rule (2.3) everywhere in the phase space. Thus we may say that the phase randomisation on the VNL releases the system from the quantum suppression, but it inevitably introduces a classical noise of quantum level, i.e. $O(\sqrt{\hbar})$.

The fact mentioned above can be verified also from an observation of the momentum distribution function. We compare in figure 7 examples of the momentum distribution functions $P(p)$ obtained for various processes. We emphasise that all the results are obtained for a single-sample process and no averaging procedures have been carried out. For the pure classical process (C1) the envelope of $P(p)$ has a typical diffusion-type distribution, i.e. $\log P(p) \propto -p^2$. The oscillatory structure of $P(p)$ is due to the existence of equally spaced resonance regions around the elliptic fixed points $(p, \theta) = (2n\pi, \pi)(n$: integer). For the pure quantum process (Q1), in contrast, the envelope of $P(p)$ decays like $\log P(p) \propto -p$, which is a manifestation of Anderson localisation.

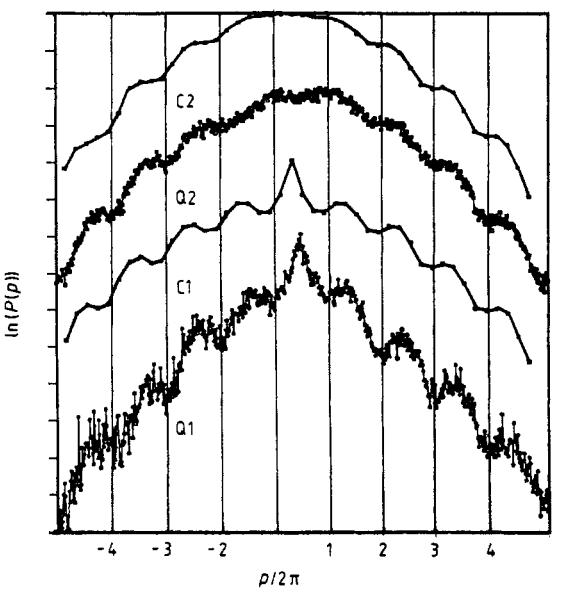

Figure 7. Momentum distribution functions $P(p) = |\langle p|\psi(t)\rangle|^2$ obtained for various processes: Q1, pure quantum process; Q2, composite (= pure quantum evolution + measurement) process; C1, pure classical process; C2, classical process driven by a classical noise of quantum level (see text).

(Q2) indicates a result for the composite process $O(t) = UR(t)$. The distribution function certainly restores the classical form, but its oscillatory structure is less pronounced compared with the pure classical result. However, if we add a classical noise to the pure classical process (C2), the $P(p)$ agrees quite well with the result of the composite process $O(t)$. The intensity of 'quantum' noise is quite small ($O(\sqrt{h})$) and is adjusted in such a way that it may reproduce the relation (4.3).

Finally we present clear evidence that the phase randomisation on the VNL restores the classical Liouville dynamics of the distribution function in the phase space. Figure 8 shows the time evolution of the pattern of the phase space distribution function due to various evolution processes: (a) the pure quantum process U, (b) the composite quantum process $O(t) = UR(t)$ and (c) the classical process with classical noise. The phase space domains covered by the symbols • and · indicate the regions of σ_P with $P = 0.9$ and 0.5, respectively. Therefore, the former represents the region with non-negligible probability level, whereas the latter shows the region in which the main probability is concentrated.

In the initial stage all the patterns of probability distributions move around 'holes' corresponding to the resonance regions filled with elliptic orbits around the fixed points $(p, \theta) = (2n\pi, \pi)(n$: integer). In ($a$) the probability distribution is confined in a region between the elliptic holes which persist forever. On the one hand, in case (b), the pattern of probability distribution goes around the elliptic holes with 'arms' stretching outward. The latter motion leads to a diffusion of probability in the direction of p, which eventually fills up the elliptic holes in the central region. This behaviour in (b)

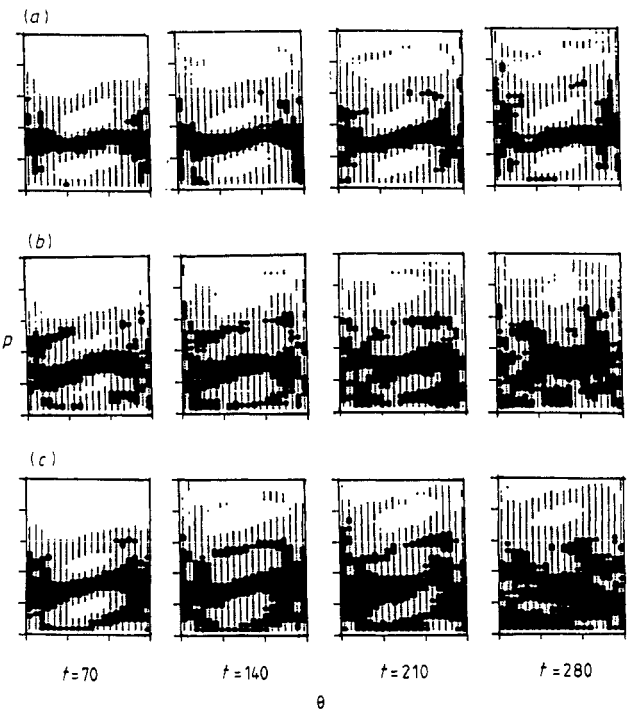

Figure 8. Time evolution of the distribution function in the phase space: (a) pure quantum process (Q1 in figure 7); (b) composite process (Q2 in figure 7); (c) classical process driven by a classical noise of quantum level (C2 in figure 7).

agrees quite well with the classical evolution depicted in (c). We emphasise again that all the results mentioned above were obtained for a single-sample process. Averaging over many sample processes does not alter the essential features of our results. This is because a phase randomisation is done on a microscopic scale everywhere in the phase space. This fact implies that the system restores the classical motion in a *single* sequence of pure evolution followed by measurement processes†.

We note that the coupling strength of the measurement system with the host system required to restore the Liouville dynamics may be quite small in the semiclassical limit $\hbar \to 0$. Indeed the condition $\nu > \nu_1$ means that the collective coupling strength $\sqrt{N}\,\zeta^{(\alpha)}$ is larger than $\nu_1/\sqrt{T} \sim O(\hbar/\log \hbar)$.

Before closing this section it should be emphasised that a recovery of the classical diffusion does not necessarily mean a restoration of the classical motion in the phase space. Whether the entire classical motion is restored or not depends upon the design of the measurement process. Indeed, we can construct an alternative measurement process on the basis of the 'squeezed' von Neumann lattice. The squeezed VNL consists of rectangular cells with different sizes $2\pi/L_p$ and $2\pi/L_\theta$ in the θ and p directions, respectively, where the condition $(2\pi)^2/L_\theta L_p = h$ should be fulfilled. Consider the ultimate limit $L_p = 1$, then the squeezed VNL basis is just the set of momentum eigenstates. Such a basis enables us to construct a measurement process for the momentum distribution function, and the back-action of such a process certainly restores something very similar to the classical chaotic diffusion; however, it fully disturbs the distribution in the θ direction and eventually spoils the classical phase space distribution to be restored.

5. Discussion

There seems to be a serious discrepancy between classical chaotic dynamics and quantum chaotic dynamics. Such a discrepancy is due to quantum interference formed everywhere in the phase space by the stretching and folding dynamics inherent in the chaotic behaviour. This discrepancy is, however, removed by incorporating the measurement processes into the pure evolution process. We have proposed a fully quantum dynamical model of a quantum non-demolition measurement for the phase space distribution function. The pure dynamical process followed by the measurement process restores the classical Liouville dynamics in the phase space. The fact that such a composite process can reproduce the classical Liouville dynamics seems to be quite instructive. We are likely to suppose that the quantum dynamics approaches the classical dynamics by only taking the mathematical limit $\hbar \to 0$. This is not the case in particular for chaotic systems. Simultaneously, letting $\hbar \to 0$, we have to make the system macroscopic. Then any dynamical perturbations due to the degrees of freedom which do not directly contribute to the chaotic motion appear and they destroy the mechanism for quantum suppression of chaos. The measurement process is an example of such a perturbation process.

† Sarkar *et al* (1987) also examined the effect of 'measurement' process on the dynamics of the quantum kicked rotor. Contrary to our result they report that the measurement process does not alter the essential feature of quantum motion. This is because their 'measurement' system is no more than a *microscopic* two-level system. Needless to say, a measurement system must be a macroscopic one composed of many degrees of freedom as proposed in the present work.

The measurement process, however, seems to have a more profound significance. In classical dynamics the measurement for the physical quantities necessary to describe the classical motion seems to be implicitly assumed. It is, therefore, quite natural that the quantum process into which the measurement process is incorporated restores the classical dynamics in any case. As emphasised in § 4, the classical dynamics thus recovered is inevitably accompanied by a classical noise of quantum level. Therefore, there is in principle no classical system which is free from the influence of noise.

In classical chaotic dynamics a small error in the initial condition is amplified continuously. This fact means that classical chaos has an ability of generating information (or entropy). However, in quantum chaos the generation of chaotic information terminates at a finite timescale. To maintain the ability of generating information we have to introduce 'information' from the external world. In our case such information is carried by the ground-state quantum fluctuation of 'atoms' constituting the measurement system. Although the amplitude of fluctuation carrying the information is quite small and of microscopic level, such a fluctuation is amplified through the mechanism of quantum chaos and finally destroys the quantum suppression mechanism to restore the ability of generating macroscopic uncertainty, i.e. the classical chaotic information. A basic question to be elucidated is in what process the microscopic information supplied by the measurement system is converted into the macroscopic chaotic information through the mechanism inherent in quantum chaos.

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References

Berry M V, Balazs N L, Tabor M and Voros A 1979 *Ann. Phys.* **122** 26
Casati G, Chirikov B V, Ford J and Izrailev F M 1979 *Stochastic Behavior in Classical and Quantum Hamiltonian Systems* (Lecture Notes in Physics **93**) ed G Casati and J Ford (Berlin: Springer) p 334
Casati G, Chirikov B V, Guarneri I and Shepelyansky D L 1986 *Phys. Rev. Lett.* **56** 2437
Chirikov B V 1979 *Phys. Rep.* **52** 265
Chirikov B V, Izrailev F M and Shepelyansky D L 1981 *Sov. Sci. Rev. Sect.* **C2** 209
Fishman S, Grempel D R and Prange R E 1982 *Phys. Rev. Lett.* **49** 509
—— 1987 *Phys. Rev.* A **36** 289
Greene J 1979 *J. Math. Phys.* **20** 1183
Ott E, Antonsen T M and Hanson J D 1984 *Phys. Rev. Lett.* **53** 2187
Sarkar S and Satchell J S 1987 *Europhys. Lett.* **4** 133
Shepelyansky D L 1983 *Physica* **8D** 208
Takahashi K and Saito N 1985 *Phys. Rev. Lett.* **55** 645
Thouless D J 1977 *Phys. Rev. Lett.* **39** 1167
Toda M and Ikeda K 1987a *Phys. Lett.* **124A** 165
—— 1987b *J. Phys. A: Math. Gen.* **20** 3833
von Neumann J 1932 *Die Mathematische Grundlagen der Quantum-mekanik* (Berlin: Springer)